J. Fluid Mech. (1970), *vol.* 43, *part* 2, *pp.* 279–290
Printed in Great Britain

The stability of pipe flow
Part 1. Asymptotic analysis for small wave-numbers

By W. P. GRAEBEL

Department of Engineering Mechanics, The University of Michigan, Ann Arbor, Michigan

(Received 26 December 1969)

The instability of Poiseuille flow in a pipe is considered for small disturbances. An asymptotic analysis is used which is similar to that found successful in plane Poiseuille flow. The disturbance is taken to travel in a spiral fashion, and comparison of the radial velocity component with the transverse component in the plane case shows a high degree of similarity, particularly near the critical point where the disturbance and primary flow travel with the same speed. Instability is found for azimuthal wave-numbers of 2 or greater, although the corresponding minimum Reynolds numbers are too small to compare favourably with either experiments or the initial restrictions on the magnitude of the wave-number.

1. Introduction

Instability of the flow of fluids in circular pipes has hitherto not been found by analytic methods. Pretch (1941), Pekeris (1948), Sexl & Spielberg (1958), and Corcos & Sellars (1959) have all considered axially symmetric disturbances and found them to be stable. Spielberg & Timan (1960) have studied the case of disturbances independent of the direction of the primary flow and have concluded that such disturbances are also stable. Lessen, Sadler & Liu (1968) have considered non-symmetric disturbances, but limited their attention to the case $n = 1$, where n is the angular wave-number. They found this case to be stable, and argued that 'physical reasoning would indicate that the first azimuthal mode ($n = 1$) should have the least stable behaviour'. The details of this physical reasoning are not presented, and the numerical technique they use for their investigation is claimed not to be suited for consideration of higher modes. Experimental results by Leite (1959) and Fox, Lessen & Bhat (1968) do indicate that pipe flow first becomes unstable due to non-axially symmetric disturbances, although measurements of n are not presented.

The present series of papers re-examines the various aspects of the problem for non-symmetric disturbances. In this first part, the governing stability equations are shown to bear a strong resemblance to those for plane Poiseuille flow, and an asymptotic technique similar to that used by Lin (1945) for plane Poiseuille flow is adapted to the present problem. The wave-numbers are assumed to be of order one. It is found that modes with values of n greater than unity are unstable, and the results agree in some respects with experimental measurements, although the critical Reynolds numbers obtained are too low to satisfy the initial order estimates of the wave-numbers.

We introduce disturbances of the form

$$u' = \frac{ir_0}{r}u(y)\,E,$$

$$v' = \frac{r}{nr_0}v(y)\,E,$$

$$w' = \frac{1}{k}w(y)\,E,$$

$$p' = i\rho W_m\left[\frac{1}{k}q(y)+\frac{2}{R}\frac{du}{dy}\right]E,$$

where $$E = \exp i(kz/r_0 - W_m kct/r_0 + n\theta),$$

$$R = W_m r_0/\nu = W_{\text{ave}}(2r_0)/\nu,$$

and $$y = (r/r_0)^2.$$

Here r_0 is the pipe radius, and W_m the primary flow speed along the centreline. The use of y rather than r leads to some mathematical simplicity and is convenient in pointing out the analogy between this problem and the stability of plane Poiseuille flow, but is not crucial in any way. The wave-numbers n (necessarily an integer) and k represent the wave-numbers measured in the azimuthal and axial directions respectively. The wave speed measured along the z axis is W_m times the real part of c. The imaginary part of c is proportional to the growth rate of the disturbance; our attention will be devoted primarily to neutral disturbances, where c is real.

Substitution of the above forms into the Navier–Stokes equations and subsequent linearization for small disturbances yields the following governing equations for the disturbance:

$$2\frac{du}{dy}+v+w = 0, \tag{1.1}$$

$$2ykR\frac{dq}{dy} = -k^2\left\{2v+u\left[\frac{n^2}{y}+k^2+ikR(W-c)\right]\right\}, \tag{1.2}$$

$$4y\frac{d^2v}{dy^2}+8\frac{dv}{dy}-\left[\frac{n^2}{y}+k^2+ikR(W-c)\right]v-2n^2\frac{u}{y^2}+2n^2\frac{du}{dy}\frac{1}{y}+\frac{n^2}{k^2}\frac{kR}{y}q = 0, \tag{1.3}$$

$$4y\frac{d^2w}{dy^2}+4\frac{dw}{dy}-\left[\frac{n^2}{y}+k^2+ikR(W-c)\right]w-2ikRu\frac{dW}{dy}+2k^2\frac{du}{dy}+kRq = 0, \tag{1.4}$$

where $W = 1-y$ is the dimensionless primary velocity. We note that n appears always as a square, and thus can be positive or negative without affecting the stability characteristics. If an analogous three-dimensional disturbance is used for plane Poiseuille flow, a striking similarity is found between the two sets of equations, particularly if the radial velocity component in the present case is compared with the transverse velocity component in the plane case.

An exact solution of equations (1.1) to (1.4) is, of course, not possible. However, since the Reynolds number can be expected to be large (albeit perhaps not as large as in the plane case), an asymptotic analysis similar to that used by Heisenberg and Lin for plane Poiseuille flow is attempted.

The correct starting point for the analysis depends upon the range of values of c, k, and n to be considered. In particular the value of c determines where order changes can occur in the solutions. These difficulties have been pointed out by Graebel (1966) and, more recently, by Eagles (1969). We adopt here the mixed approximation used by Heisenberg and Lin in plane flows, in an attempt to obtain a first estimate on the various parameters. The need for an approximation valid for large values of n and particularly k will be shown.

Solutions will first be found valid everywhere except at the critical point $y_c = 1-c$, where the disturbances wave and the primary flow travel at the same speed. These solutions will be referred to as the outer solutions, and the corresponding region will be labelled the outer region. Solutions valid near y_c will then be obtained; these are naturally referred to as inner solutions and the corresponding region as the inner region. Composite solutions could be obtained by the addition of the inner and outer solutions, and subsequent removal of duplicated terms.

2. The outer solution

We first seek the inviscid solutions. The correct independent variable appropriate to solutions which are neither exponentially large or small is y; all disturbance quantities will then be of the same order. Letting

$$u \sim \bar{u}(y,\epsilon) = \sum_{m=0} \mathscr{E}_m(\epsilon)\,\bar{u}_m(y)$$

and likewise for the remaining dependent variables, the governing differential system is found to be

$$2y\frac{dq_0}{dy} + ik^2(W-c)\,\bar{u}_0 = 0,$$

$$n^2\bar{q}_0 - iyk^2(W-c)\,\bar{v}_0 = 0,$$

$$\bar{q}_0 - i(W-c)\,\bar{w}_0 - 2i\bar{u}_0\frac{dW}{dy} = 0,$$

$$2\frac{d\bar{u}_0}{dy} + \bar{v}_0 + \bar{w}_0 = 0.$$

This can be rearranged to give

$$\bar{v}_0 = \frac{2n^2}{n^2+yk^2}\left[\bar{u}_0\frac{dW/dy}{W-c} - \frac{du_0}{dy}\right], \tag{2.1}$$

$$\bar{w}_0 = \frac{-2n^2}{n^2+yk^2}\left[\bar{u}_0\frac{dW/dy}{W-c} + \frac{yk^2}{n^2}\frac{d\bar{u}_0}{dy}\right], \tag{2.2}$$

$$\bar{q}_0 = iyk^2(W-c)\,\bar{v}_0/n^2, \tag{2.3}$$

where $\bar{u}_0$ is determined from

$$y^2\left(1+\frac{yk^2}{n^2}\right)(W-c)\frac{d^2\bar{u}_0}{dy^2}+y(W-c)\frac{d\bar{u}_0}{dy}$$
$$+\left[y-0{\cdot}25n^2(W-c)\left(1+\frac{yk^2}{n^2}\right)^2\right]\bar{u}_0 = 0. \quad (2.4)$$

Thus $\bar{u}_0$ has a logarithmic singularity at the point where $W = c$ (i.e. where $y = y_c \equiv 1-c$), plus poles (of order n in r) at $y = 0$ and at $y = -n^2/k^2$. Solving by the method of Frobenius, two solutions can be found near y_c of the form

$$\phi_1(y) = (y-y_c)\frac{k^2}{n^2}+\frac{k^2(n^2+y_ck^2)}{24n^2y_c^2}(y-y_c)^3+\ldots,$$

$$\phi_2(y) = \phi_1(y)\ln\left[(y-y_c)\frac{k^2}{n^2}\right]+\frac{k^2}{n^2}y_c\left(1+\frac{k^2}{n^2}y_c\right)+\frac{(y-y_c)^2}{k^2y_c}\left[-n^2+\frac{(n^2+k^2y_c)^2}{8}\right]+\ldots.$$

Near $y = 0$, the non-singular solution has the form

$$\phi_3(y) = y^{\frac{1}{2}n}\left\{1+\frac{y}{[y_cn^2(n+1)]}[y_ck^2(\tfrac{1}{4}n^2+\tfrac{1}{2}n)-n^2]\right\}+\ldots.$$

We write $\Phi_1(y)$ and $\Phi_2(y)$ as the solutions of (2.4) analytic everywhere except possibly y_c, 0, or $-n^2/k^2$, and which behave like $\phi_1(y)$ and $\phi_2(y)$ near y_c. $\Phi_3(y)$ is the combination of $\Phi_1(y)$ and $\Phi_2(y)$ which is proportional to $\phi_3(y)$ near $y = 0$. Letting β be an arbitrary positive number less than n^2/k^2, then

$$\Phi_3(y) = [\phi_3(\beta)\phi_2'(\beta)-\phi_2(\beta)\phi_3'(\beta)]\,\Phi_1(y)+[\phi_1(\beta)\phi_3'(\beta)-\phi_3(\beta)\phi_1'(\beta)]\,\Phi_2(y). \quad (2.5)$$

Since for $c \geqslant 0{\cdot}5$, ϕ_1 and ϕ_2 will likely not converge at $y = 1$, Φ_1 and Φ_2 are the solutions to be matched to the inner solution, and their form has to be found by suitable expansions about $y = 1$. The details, being straightforward, are omitted. The condition that the solutions be finite at $y = 0$ will dictate that the necessary combination of Φ_1 and Φ_2 in the solution is, in fact, proportional to Φ_3, which will give us the characteristic equation.

The previous approach gives only those outer expansions which do not either grow or decay exponentially as a function of some non-zero power of the Reynolds number. For the purpose of estimating numerical accuracy, it is desirable to have as well the outer expansions of those solutions which are of exponential character. These can easily be found by the WKB method. That is, a solution is sought in the form (where f is any of the dependent variables)

$$\sum_{m=0}^{M}\lambda^{-m}f_m(y)\exp\left(\lambda\int g\,dy\right),$$

the f_m's, g and λ being determined by the differential system. The result is easily found to give $\lambda^2 = kR$ and $g^2 = \frac{1}{4}i(-1+y_c/y)$. (This approach is also suited for finding the two outer expansions with first terms Φ_1 and Φ_2; the corresponding value for g is zero.) The velocities and pressure so determined are given to lowest order in table 1, matched to the inner solutions found later and presented in table 2. V_0 and W_0 are constants (functions of λ) which would be determined by higher-order matching. Two further outer expansions which grow exponentially are easily found by using the negative square root for g in solutions 3 and 4; these would merge with the inner solution corresponding to the Hankel function of second kind. These are asymptotic expansions of those solutions singular at the origin and hence not of interest in the present work.

It is seen that solutions 2, 3 and 4 all have branch points at the critical point, and continuation through this point is not *a priori* obvious. The method used is essentially a comparison of the given system with one having constant coefficients; it is not surprising that this breaks down when one of the coefficients changes sign. We next adopt an approach which compares the given system with one having linear coefficients, the comparison providing the needed continuation through the critical point.

3. The inner solution

To continue the outer solutions across the critical point, notice must be taken that the relative velocity between the primary flow and the disturbance is small in the neighbourhood of the critical point, and the order of the scaling will vary. As in plane Poiseuille flow, the correct order is $\epsilon = (kR)^{-\frac{1}{3}}$. Then solutions 3 and 4 will decay exponentially going from the critical point to the interior of the pipe, and grow exponentially going from the critical point to the wall. The wall effect is, as seen from the outer solutions, of order $(kR)^{-\frac{1}{2}}$. If $c = O(\epsilon)$, then the two regions clearly overlap and the inner solution will dominate, since it represents a thicker region. For increasingly larger values of c (smaller y_c), however, the region associated with the critical point moves from the wall and will eventually be distinct from the wall region. The appropriate variable for the wall region is $\eta = (y_c - y)/\epsilon$. When the two regions overlap, this is also the appropriate variable for the wall region, for a region with thickness of order ϵ will dominate one with thickness of order $\epsilon^{\frac{3}{2}}$. When, however, the regions become distinct, the appropriate wall variable is $(1-y)/\epsilon^{\frac{3}{2}}$, as shown in Graebel (1966). Such a choice, however, introduces the necessity of considering the cases of various orders of magnitude of c as being distinct and requiring separate analyses. In the following, we will use the independent variable η for all $\delta_1 \leqslant y_c < 1$, $\delta_1 > O(\epsilon^{\frac{3}{2}})$, and then comment on the numerical accuracy which can be expected for this range of y_c.

We let

$$u \sim u^*(\eta, \epsilon) = \sum_{m=0} \epsilon^{m+1} u^*_m(\eta),$$

$$v \sim v^*(\eta, \epsilon) = \sum_{m=0} \epsilon^{m} v^*_m(\eta),$$

$$w \sim w^*(\eta, \epsilon) = \sum_{m=0} \epsilon^{m} w^*_m(\eta),$$

and

$$q \sim q^*(\eta, \epsilon) = \sum_{m=0} \epsilon^{m+1} q^*_m(\eta).$$

The difference in ordering is dictated by continuity and the wish to have the equations coupled to the maximum degree, and is analogous to the ordering in boundary-layer theory. Substitution of these into the stability equations (1.1) to (1.4) gives

$$\frac{dq^*_0}{d\eta} = 0, \quad \frac{dq^*_1}{d\eta} = 0, \qquad (3.1), (3.2)$$

$$4y_c\frac{d^2v^*_0}{d\eta^2}-i\eta v^*_0+\frac{n^2q^*_0}{k^2y_c} = 0, \tag{3.3}$$

$$4y_c\frac{d^2v^*_1}{d\eta^2}-i\eta v^*_1+\frac{n^2q^*_1}{k^2y_c}+\frac{\eta n^2q^*_0}{k^2y_c}-4\eta\frac{d^2v^*_0}{d\eta^2}-8\frac{dv^*_0}{d\eta} = 0, \tag{3.4}$$

$$4y_c\frac{d^2w^*_0}{d\eta^2}-i\eta w^*_0+2iu^*_0+q^*_0 = 0, \tag{3.5}$$

$$4y_c\frac{d^2w^*_1}{d\eta^2}-i\eta w^*_1+2iu^*_1+q^*_1-4\eta\frac{d^2w^*_0}{d\eta^2}-4\frac{dw^*_0}{d\eta} = 0, \tag{3.6}$$

$$2\frac{du^*_0}{d\eta}-v^*_0-w^*_0 = 0, \tag{3.7}$$

$$2\frac{du^*_1}{d\eta}-v^*_1-w^*_1 = 0, \tag{3.8}$$

for the first two orders of ϵ. These show that q^*_0 is a constant. Its magnitude is arbitrary, for since the problem is homogeneous, the solution is indeterminate to a scale factor: it is convenient to choose

$$q^*_0 = 2ik^2\eta_0(1-F)/(k^2+n^2/y_c),$$

where
$$\eta_0 = (y_c-1)/\epsilon, \quad F = \chi(\eta_0)/\eta_0\chi'(\eta_0),$$

and
$$\chi(\eta) = \int_\infty^\eta ds\int_\infty^s d\sigma\, \sigma^{\frac{1}{2}}H^{(1)}_{\frac{1}{3}}\left[\tfrac{2}{3}i\left(\frac{i\sigma^3}{4y_c}\right)^{\frac{1}{2}}\right].$$

To ensure correct behaviour to match to the outer solution (i.e. exponential decay as $\eta\to\infty$), it is necessary that $-7\pi/6 < \arg\eta < \pi/6$. If we restrict attention first to the case where c is very small, the solutions satisfying the boundary conditions at the wall are found to be

$$u^*_0(\eta) = \eta+\eta_0(F-1)-\chi(\eta)/\chi'(\eta_0), \tag{3.9}$$

$$v^*_0(\eta) = (i/4y_c)^{\frac{1}{3}}\, 2\eta_0 n^2(1-F)/(n^2+k^2y_c) * \{G(\eta)-\chi''(\eta)\, G(\eta_0)/\chi''(\eta_0)\}, \tag{3.10}$$

$$w^*_0(\eta) = -v_0(\eta)+2-2\chi'(\eta)/\chi'(\eta_0), \tag{3.11}$$

where
$$G(\eta) = i\int_0^\infty ds\exp -i[\eta(i/4y_c)^{\frac{1}{3}}s+s^3/3],$$

and
$$\begin{aligned} u^*_1(\eta) = {} & \frac{q^*_1}{q^*_0}u^*_0(\eta)+\frac{1}{5y_c}\left[\eta^2u^{*\prime}_0(\eta)-2\int_{\eta_0}^\eta u^*_0(x)\,dx\right]-\tfrac{2}{3}iv^{*\prime}_0(\eta)+A_1\chi(\eta)+B_1\eta \\ & -\frac{i}{2}\frac{n^2}{k^2}\frac{q^*_0}{y_c^2}\left(\frac{i}{4y_c}\right)^{\frac{1}{3}}\int_{\eta_0}^\eta dx\int_{\eta_0}^x G(t)\,dt-0{\cdot}8iu^{*\prime\prime}_0(\eta_0) \\ & +\left(\frac{iq^*_0}{y_c}\right)\left\{-0{\cdot}1\eta_0+\frac{n^2}{k^2}\left[0{\cdot}4\frac{\eta_0}{y_c}-2i\left(\frac{i}{4y_c}\right)^{\frac{1}{3}}G'(\eta_0)\right]\right\}, \end{aligned} \tag{3.12}$$

$$\begin{aligned} v^*_1(\eta) = {} & \frac{q^*_1}{q^*_0}v^*_0(\eta)-0.4i\frac{n^2q^*_0}{k^2y_c}\left[1-\frac{\chi''(\eta)}{\chi''(\eta_0)}\right]+\frac{0.8}{y_c}\eta v^*_0(\eta) \\ & +\frac{\eta^2}{5y_c}v^{*\prime}_0(\eta)-\frac{\eta_0^2v^{*\prime}_0(\eta_0)\chi''(\eta)}{5y_c\chi''(\eta_0)} \end{aligned} \tag{3.13}$$

$$w^*_1(\eta) = -v^*_1(\eta)+2u^{*\prime}_1(\eta). \tag{3.14}$$

A_1, B_1 are constants chosen to satisfy the conditions $u^*_1(\eta_0) = u^{*\prime}_1(\eta_0) = 0$. The quantity q^*_1 is again arbitrary for the same reason as was q^*_0. It can conveniently be chosen so that the coefficient multiplying the term linear in η in the asymptotic expansion of $\bar{u}_1$ as $n \to +\infty$ is zero.

For the purpose of matching to the outer expansion, it is necessary to know the behaviour of $u^*(\eta)$ as $\eta \to +\infty$. The χ's all decay exponentially, and the only quantity requiring particular attention is the function G. We note that, by simple integration by parts,

$$\int_{\eta_0}^{\eta} dx \int_{\eta_0}^{\eta} G(t)\,dt = \eta \int_{\eta_0}^{\eta} G(x)\,dx + \left[\frac{\eta}{a} - \frac{1}{a^3}\frac{dG(\eta)}{d\eta}\right]_{\eta_0}^{\eta},$$

where $a = (i/4y_c)^{\frac{1}{3}}$. For the arguments in question,

$$G(\eta) \to -\frac{1}{a\eta}$$

and
$$\int_0^{\eta} G(x)\,dx \to \frac{1}{a}\left[-\ln(3^{\frac{1}{3}}\eta\,|a|) - \tfrac{2}{3}\gamma + \frac{i\pi}{6}\right] + O\left(\frac{1}{\eta}\right).$$

Thus
$$u^*(\eta,\epsilon) \to \epsilon[\eta + \eta_0(F-1)] + \epsilon^2\left[\frac{n^2\eta_0(1-F)}{y_c(n^2+k^2y_c)}\,\eta\ln\eta + O(1)\right] + O(\epsilon^3).$$

The combination of outer solutions which match this is

$$\bar{u}(y,\epsilon) = -\frac{n^2}{k^2}\phi_1(y) + \frac{n^4\eta_0\epsilon(F-1)}{k^2y_c(n^2+k^2y_c)}\left[\phi_2(y) - \phi_1(y)\ln\left(-\epsilon\frac{k^2}{n^2}\right)\right]. \quad (3.15)$$

From the above asymptotic forms, it is clear that the function χ has the steep behaviour which one usually associates with the wall layer, and the function G is taking care of the singularity which the inviscid solution exhibits at the critical point. The four non-exponentially growing independent solutions in the inner region are presented to the first two orders in table 2. (The axial component of velocity is found from $w = -v + 2\,du/d\eta$.) If one takes the asymptotic form of χ and writes it in terms of y, the solutions of the form $(1-y)(kR)^{\frac{1}{2}}$ presented in Graebel (1966) immediately appear. Solution 2 is appropriate to the region near the critical point, and must be the continuation of the inviscid solution appropriate to the critical region. Similarly solution 1 is the continuation of ϕ_1, and is appropriate to all steep regions. These results are summarized in tables 1 and 2.

While the characteristic equation could be obtained from equation (3.15) by imposing the finiteness condition at the origin, the limitation of small c is too restrictive to provide numerical accuracy. Even in plane Poiseuille flow one has to go to one higher order of approximation in that part of the inner solution corresponding to the constant in ϕ_2, since this will bring in the needed imaginary term to keep the inviscid effects from being purely real. To do this, we recognize that the first two terms in equation (3.9) are inner expansions of Φ_1 and Φ_2. Introducing these in place of the linear terms gives

$$u^*_0 = \{[F(y_c-1)\,\Phi_2'(1) + \Phi_2(1)]\,\Phi_1(y) \\ - [\Phi_1(1) + F(y_c-1)\,\Phi_1'(1)] * \Phi_2(y)\}/\epsilon\Delta - \chi(\eta)/\chi'(\eta_0),$$

with Δ the Jacobian of Φ_1 and Φ_2 evaluated at $y = 1$. This solution must be proportional to Φ_3 near the origin to satisfy conditions of finiteness there. Comparing the above with the earlier expression for Φ_3 in terms of Φ_1 and Φ_2 gives the characteristic equation

$$[F(y_c-1)\,\Phi_2'(1)+\Phi_2(1)]/[\Phi_1(1)+F(y_c-1)\,\Phi_1'(1)] = S,$$

where $$S = [\phi_3(\beta)\,\phi_2'(\beta)-\phi_2(\beta)\,\phi_3'(\beta)]/[\phi_3(\beta)\,\phi_1'(\beta)-\phi_1(\beta)\,\phi_3'(\beta)].$$

It is convenient to rewrite this in the form

$$(1-F)^{-1} = 1/\{1+[\Phi_2(1)-S\Phi_1(1)]/(y_c-1)\,[\Phi_2'(1)-S\Phi_1'(1)]\}, \tag{3.16}$$

where now the Reynolds number appears only on the left-hand side of the equation.

	u	v	w	q
1	$-\dfrac{\epsilon n^2}{k^2}\phi_1$	eqn. (2.1)	eqn. (2.2)	eqn. (2.3)
2	$\dfrac{n^4\phi_2}{k^2y_c(n^2+k^2y_c)}$	eqn. (2.1)	eqn. (2.2)	eqn. (2.3)
3	$\dfrac{\epsilon^{\frac{5}{4}}\mathscr{E}}{yg^{\frac{5}{2}}}$	$\dfrac{\mathscr{E}}{\lambda}\left\{\dfrac{-\epsilon^{\frac{1}{4}}}{yg^{\frac{1}{2}}}+\dfrac{n^2W_0}{yy_c^2g^{\frac{7}{2}}}\times(\tfrac{2}{3}-4ig^2-16g^4)\right\}$	λu	$-\dfrac{2k^2gu}{\lambda}$
4	$-(v+w)/2g\lambda$	$-\dfrac{\epsilon^{\frac{1}{4}}\mathscr{E}}{yg^{\frac{1}{2}}}$	$\dfrac{\mathscr{E}}{2yg^{\frac{3}{2}}i^{\frac{1}{2}}}\{W_0+\tfrac{1}{2}\epsilon^{\frac{1}{4}}\times\ln[y(1+4ig^2+4gi^{\frac{1}{2}}]\}$	$-\dfrac{k^2}{\lambda^2}(v+w)$

TABLE 1. Outer expansions of four independent solutions correct to one term.
$\lambda = (kR)^{\frac{1}{2}}, \quad g = [\tfrac{1}{4}i(-1+y_c/y)]^{\frac{1}{2}}, \quad \mathscr{E} = y_c(3/\pi i)^{\frac{1}{2}}\exp[\lambda\int g\,dy-\tfrac{5}{12}\pi i]$

	u	v	q
1	$\epsilon\eta$	$o(\epsilon)$	$o(\epsilon^2)$
2	$\epsilon+\epsilon^2\left[i(n^2+k^2y_c)\,q_1/2k^2y_c+\dfrac{n^2}{n^2+k^2y_c}\left(\dfrac{i}{4y_c}\right)^{\frac{1}{3}}\times\left(\tfrac{4}{3}i\dfrac{dG(\eta)}{d\eta}-\dfrac{1}{y_c}\displaystyle\int_0^\eta dx\int_0^x G(t)\,dt\right)\right]$	$-\dfrac{2n^2}{n^2+k^2y_c}\left\{\left(\dfrac{i}{4y_c}\right)^{\frac{1}{3}}G(\eta)\times\left[1-\dfrac{\epsilon iq_1(n^2+k^2y_c)}{2k^2y_c}+\dfrac{4\epsilon\eta}{5y_c}\right]+n^2\epsilon\left(\dfrac{i}{4y_c}\right)^{\frac{1}{3}}\dfrac{dG(\eta)}{d\eta}-0{\cdot}4\epsilon\right\}$	$-\dfrac{2ik^2y_c\epsilon}{n^2+k^2y_c}+\epsilon^2q_1$
3	$\epsilon\chi(\eta)+\dfrac{\epsilon^2}{5y_c}\left(\eta^2\dfrac{d\chi(\eta)}{d\eta}-2\displaystyle\int_\infty^\eta\chi(t)\,dt\right)$	$o(\epsilon)$	$o(\epsilon^2)$
4	$\tfrac{2}{3}i\epsilon^2\dfrac{d^3\chi(\eta)}{d\eta^3}$	$-\dfrac{d^2\chi(\eta)}{d\eta^2}\left(1+\dfrac{4\epsilon\eta}{5y_c}\right)-\dfrac{\epsilon\eta^2}{5y_c}\dfrac{d^3\chi(\eta)}{d\eta^3}$	$o(\epsilon^2)$

TABLE 2. Inner expansions of four independent solutions given in table 1, correct to two terms. The axial velocity component is found from $w = -v+2du/d\eta$

4. Results and discussion

The calculation of the eigenvalues based on (3.16) is straightforward if somewhat tedious. The various inviscid solutions are easily computed by developing them in Taylor series in y, $y-y_c$, or $y-1$, depending on the region of interest, and then using the recursion relations so obtained to evaluate them as functions of k, n, and y_c. It is simple to show that the imaginary part of the right-hand side of (3.16) is always positive. Tables of the left-hand side of (3.16) can be found in Lin (1945) or, more detailed, in Miles (1960). The left-hand side of (3.16) is solely a function of the parameter $\zeta = (1-y_c)(kR/4y_c)^{\frac{1}{3}}$. Upon plotting the real and imaginary part of each side of the characteristic equation as the ordinate and abscissa, the intersection of the two curves gives the required eigenvalues.

The computations were carried out for integer values of n between 1 and 10, and for $0{\cdot}1n \leqslant k \leqslant 2{\cdot}5n$. Slowness of convergence of the various series dictated the upper limit, and there seemed no practical reason to go below the lower limit, since the solutions changed very slowly with k below this range. The wave speed c was varied between 0·1 and 0·9. For $n = 1$, no eigenvalues were found, thus confirming the stability found by Lessen, Saddler & Liu (1968). For $n \geqslant 2$, the results are shown in figure 1. (Values of n greater than 5 are omitted from the various figures since they give values of the critical Reynolds number which are higher at a given k than those found for lower n's.) It was found in all cases that the only intersection of the curves occurred when the real and imaginary parts of each side of the equation were essentially 2·3 and 0, respectively; ζ then is approximately 2·3 also. The dashed lines connect points having the same wave speeds. It is seen that $n = 2$ yields the lowest critical Reynolds number, although it was not possible to determine the actual minimum value, if in fact such a minimum exists. The constancy of ζ occurs because when the values of n, k^2, and y_c are such that the real part of the right-hand side of equation (3.16) lies between 1 and 2·3, the corresponding imaginary part is to all intents and purposes, zero.

The values of the minimum Reynolds number are patently too low when compared with any experimental results. To determine the sensitivity of the present answers to the parameter c and the order of the approximations, two simple alternate computations were tried. In the first, the expression for ζ was replaced by $(1-(1-c)^{\frac{1}{2}})(2kR(1-c)^{\frac{1}{2}})^{\frac{1}{3}}$. This is the result found if the co-ordinate r rather than $y = r^2$ were used. The result is shown in figure 2. Here all curves including $n = 2$ show minimum values of the critical Reynolds number, and the corresponding minimum is higher than that given in figure 1. (The two values of ζ can be shown to agree for very small c by suitable use of the binomial theorem.)

In figure 3, the expression used for ζ was $(1-y_c)(\frac{1}{4}kR)^{\frac{1}{3}}$, the rationale here being to attempt to evaluate to a limited degree the influence of c on the approximation to the coefficients in the governing equation. This value of ζ would be obtained if in approximating (in the inner region) the coefficients of the highest derivative in equation (1.3) and (1.4), one used 4 instead of $4y_c$, i.e. if y were expanded about 1 rather than y_c. The results are qualitatively much the same as obtained from the approximation used in figure 2. The minimum critical

Reynolds number is higher than given by figure 1, and even higher than given by figure 2 for all n except $n = 2$.

While these results do not in any way give a minimum critical Reynolds number which agrees with experimental results, there are reassuring features

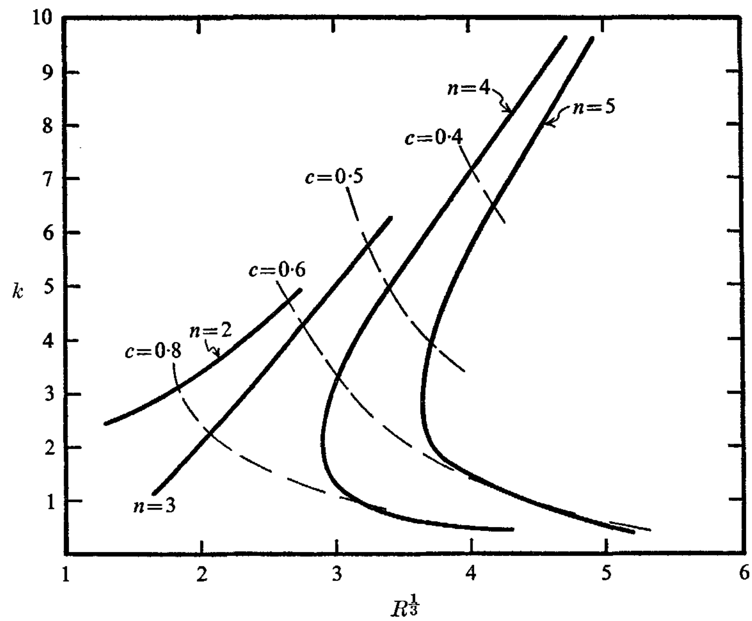

FIGURE 1. Neutral stability curves based on $\zeta = (1-y_c)\,(kR/4y_c)^{\frac{1}{3}}$.

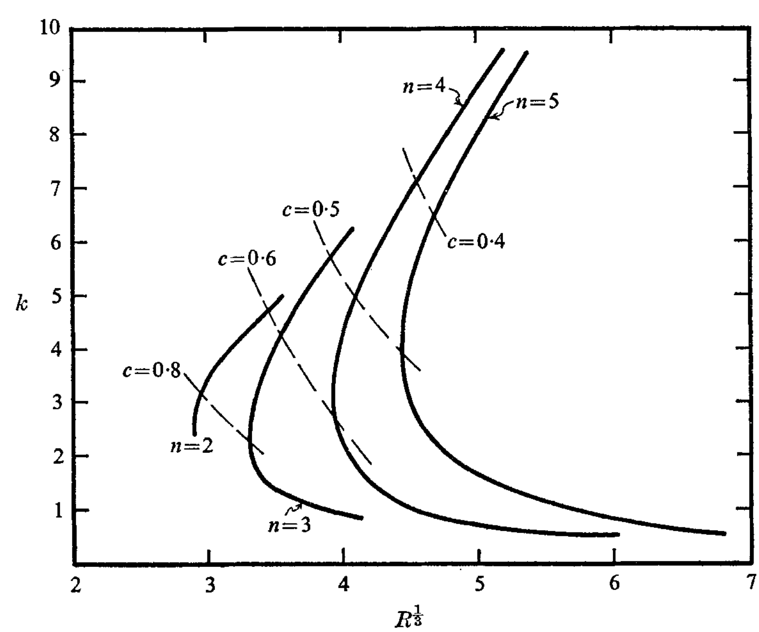

FIGURE 2. Neutral stability curves based on $\zeta = (1-(1-c)^{\frac{1}{2}})\,(2kR(1-c)^{\frac{1}{2}})^{\frac{1}{3}}$.

which indicate that a linearized theory may be valid. Fox, Lessen & Bhat (1968) indicate a minimum critical Reynolds number of approximately $R^{\frac{1}{3}} = 13$, corresponding to $c = 0{\cdot}6$ and $k = 9$. The present results are giving higher values for both c and k than are found in the plane Poiseuille case. As was pointed out initially, the present results are only valid for small n and k; we are encountering flow instability in our approximation only where k particularly is of the same order as the Reynolds number. Eagles (1969) has pointed out that the use of the present type of approximation will likely predict the correct trend, but also can be expected to lead to inaccuracies. The reason that Lin's results are as accurate as they are is precisely because for plane Poiseuille flow, the maximum values

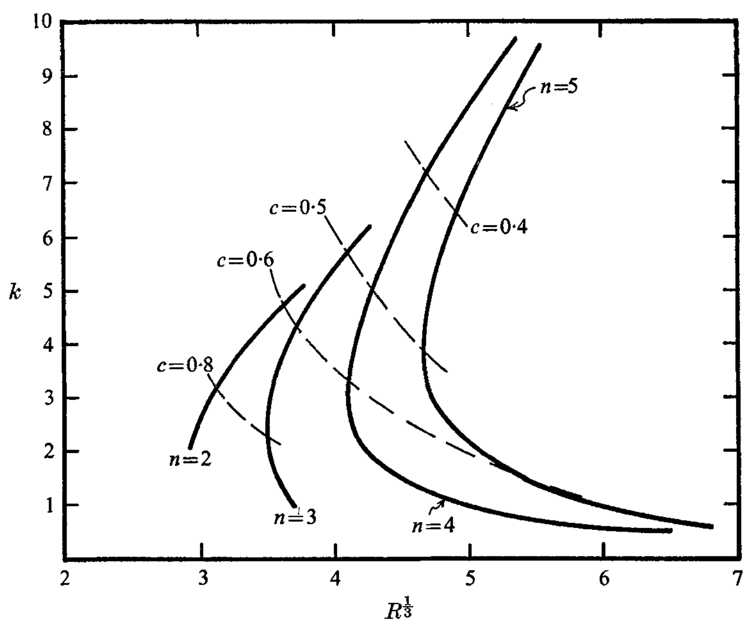

FIGURE 3. Neutral stability curves based on $\zeta = (1-y_c)\,(kR/4)^{\frac{1}{3}}$.

of c and k are much smaller than found in the present case ($0{\cdot}3$ and $1{\cdot}2$ respectively). Because of the particular way in which n and k appear in the inviscid equation (2.4), the inviscid solutions, and in particular the real parts, grow rapidly as n/k becomes small. Thus the effect of these large wave-numbers is to influence all parts of the approximation in substantial ways.

How can the present calculations be improved in accuracy? It is unlikely that carrying the present approximation to one or two higher order terms will be sufficient. In the inner region, the terms k^2+n^2/y_c which accompany the inertia terms in equations (1·2)–(1·4) will enter at the third approximation and will contribute terms which are of the order of $\epsilon^2(k^2+n^2/y_c)$ compared with the first-order terms. For values of n, k, and R of the magnitude found in the present approximation, these terms would be nearly of order one, and the convergence would appear to be slow, although no attempt was made to carry out such computations. Use of the outer expansion, or of the composite expansion suggested by Eagles (1969), do not on the surface appear fruitful for the same reason. It seems preferable, and perhaps necessary, to include such terms in the lowest-order approximation to gain the greatest accuracy with the fewest number of terms.

This research was partially sponsored by the Fluid Dynamics Branch of the Office of Naval Research under contract Nonr-1224(49) with The University of Michigan.

REFERENCES

CORCOS, G. M. & SELLARS, J. R. 1959 On the stability of fully developed flow in a pipe. *J. Fluid Mech.* **5**, 97–112.

EAGLES, P. M. 1969 Composite series in the Orr–Sommerfeld problem for symmetric channel flow. *Quart. J. Mech. Appl. Math.* **22**, 129–182.

FOX, J. A., LESSEN, M. & BHAT, W. V. 1968 Experimental investigation of the stability of Hagen–Poiseuille flow. *Phys. Fluids,* **11**, 1–4.

GRAEBEL, W. P. 1966 On determination of the characteristic equations for the stability of parallel flows. *J. Fluid Mech.* **24**, 497–508.

LEITE, R. J. 1959 An experimental investigation of the stability of Poiseuille flow. *J. Fluid Mech.* **5**, 81–96.

LESSEN, M., SADLER, S. G. & LIU, T.-Y. 1968 Stability of pipe Poiseuille flow. *Phys. Fluids,* **11**, 1404–1409.

LIN, C. C. 1945 On the stability of two-dimensional parallel flows. *Quart. Appl. Math.* **3**, 117–142.

MILES, J. W. 1960 The hydrodynamic stability of a thin film of liquid in uniform shearing motion. *J. Fluid Mech.* **8**, 593–610.

PEKERIS, C. L. 1948 Stability of the laminar flow through a straight pipe of circular cross-section to infinitesimal disturbances which are symmetrical about the axis of the pipe. *Proc. Nat. Acad. of Sci., Wash. D.C.* **34**, 285–295.

PRETCH, J. 1941 Über die Stabilität einer Laminarströmung in einem geraden Rohr mit kreisförmigem Querschnitt. *Z. angew. Math. Mech.* **21**, 204–217.

SEXL, T. & SPIELBERG, K. 1958 Zum Stabilitätsproblem der Poiseuille–Strömung. *Acta Phys. Austriaca,* **12**, 9–28.

SPIELBERG, K. & TIMAN, H. 1960 On three- and two-dimensional disturbances of pipe flow. *J. Appl. Mech.* **27**, 381–389.